KNOWLEDGEABLE RESEARCH (KR)
ISSN: 2583-6633 | CODEN: KRABAU
International Open-Access Peer-Reviewed Refereed Multidisciplinary Journal
https://knowledgeableresearch.com
Vol. 5, Issue 07,2026

Received: 07/06/2026 | Accepted: 26/07/2026 | Published: 30/07/2026

A Study on Generalized Closed Sets and Their Properties in Topological Spaces

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Abstract

The notion of generalized closed sets, first introduced by Levine, occupies a central place in the study of general topology, since it provides a natural bridge between classical closed sets and several weaker forms of closedness studied subsequently in the literature. In this paper, we present a systematic study of generalized closed sets (g-closed sets) and generalized open sets (g-open sets) in topological spaces. We recall the fundamental definitions, establish several characterizing theorems, examine the relationship of g-closed sets with other well known classes of sets such as semi-closed, pre-closed and α-closed sets, and investigate their behaviour under subspace topology and continuous mappings. Suitable examples are provided to justify that the inclusions among these classes of sets are, in general, strict. The paper concludes with a discussion of generalized continuity and some open problems for further research.

Keywords: *Topological space; closed set; generalized closed set; generalized open set; semi-open set; pre-open set; g-continuous function.*

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1. Introduction

General topology provides the abstract framework within which notions of nearness, continuity and convergence are studied without reference to a metric. Among the many generalizations of the closed set that have enriched this framework, the concept of a generalized closed set (briefly, g-closed set), introduced by N. Levine in 1970, has proved to be one of the most fruitful. A g-closed set need not be closed in the usual sense, yet it retains enough structure to allow meaningful generalizations of continuity, compactness and separation axioms.

Following Levine's original work, a large number of related notions have been introduced and studied, including semi-generalized closed sets, generalized semi-closed sets, α-generalized closed sets and generalized pre-closed sets, each obtained by weakening the closure operator involved in the defining condition. These classes together form a rich lattice of set-theoretic notions that refine our understanding of topological structure and have found applications in digital topology, the theory of ordered sets and computer science.

The purpose of the present paper is twofold. First, we collect and organize the basic theory of g-closed and g-open sets, presenting proofs of the principal characterizing results in a self-contained manner. Second, we examine how g-closed sets interact with subspaces and with continuous functions, and we use this study to motivate the notion of generalized continuity. Throughout, (X, τ) and (Y, σ) denote topological spaces with no separation axioms assumed unless explicitly stated.

2. Preliminaries

In this section we recall some definitions and known results that will be used in the sequel. Let (X, τ) be a topological space and let A be a subset of X. The closure and interior of A relative to τ are denoted by cl(A) and int(A) respectively.

Definition 2.1. A subset A of a topological space (X, τ) is said to be:

(i) semi-open if $A \subseteq cl(int(A))$, and semi-closed if $int(cl(A)) \subseteq A$;

(ii) pre-open if $A \subseteq int(cl(A))$, and pre-closed if $cl(int(A)) \subseteq A$;

(iii) α-open if $A \subseteq int(cl(int(A)))$, and α-closed if $cl(int(cl(A))) \subseteq A$;

(iv) regular open if $A = int(cl(A))$, and regular closed if $A = cl(int(A))$.

The collection of all semi-open (respectively pre-open, α-open) subsets of X is denoted by SO(X) (respectively PO(X), αO(X)). Complements of the sets above give the corresponding closed classes, denoted SC(X), PC(X) and αC(X).

Definition 2.2 (Levine, 1970). A subset A of a topological space (X, τ) is called a generalized closed set (g-closed set) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X. The complement of a g-closed set is called a generalized open set (g-open set).

Equivalently, A is g-open if and only if $F \subseteq int(A)$ whenever F is closed and $F \subseteq A$. The family of all g-closed subsets of X is denoted GC(X), and the family of all g-open subsets of X is denoted GO(X).

Definition 2.3. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be g-continuous if $f^{-1}(V)$ is a g-closed set in X for every closed set V in Y.

3. Basic Properties of Generalized Closed Sets

Theorem 3.1. Every closed set in a topological space (X, τ) is g-closed.

Proof. Let A be closed in X and suppose $A \subseteq U$ where U is open. Since A is closed, $cl(A) = A$, and $A \subseteq U$ by hypothesis, so $cl(A) \subseteq U$. Hence A is g-closed. ■

The converse of Theorem 3.1 is false in general, as the following example shows.

Example 3.2. Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{a\}, X\}$. The set $A = \{a, b\}$ is not closed in X, since its complement $\{c\}$ is not open. However, the only open set containing A is X itself, and $cl(A) = X \subseteq X$. Hence A is g-closed but not closed.

Theorem 3.3. If A is a g-closed subset of (X, τ) and $A \subseteq B \subseteq cl(A)$, then B is also g-closed.

Proof. Let U be an open set with $B \subseteq U$. Since $A \subseteq B$, we have $A \subseteq U$, and because A is g-closed, $cl(A) \subseteq U$. Also $B \subseteq cl(A)$ implies $cl(B) \subseteq cl(cl(A)) = cl(A) \subseteq U$. Thus $cl(B) \subseteq U$, and B is g-closed. ■

Theorem 3.4. A subset A of (X, τ) is g-closed if and only if $cl(A) - A$ contains no non-empty closed set.

Proof. Suppose A is g-closed and let F be a closed subset of $cl(A) - A$. Then $A \subseteq X - F$, and since F is closed, $X - F$ is open. As A is g-closed, $cl(A) \subseteq X - F$, that is, $F \subseteq X - cl(A)$. But $F \subseteq cl(A)$ as well, so $F \subseteq cl(A) \cap (X - cl(A)) = \emptyset$, whence $F = \emptyset$.

Conversely, suppose $cl(A) - A$ contains no non-empty closed set, and let U be open with $A \subseteq U$. If $cl(A)$ is not contained in U, then $cl(A) \cap (X - U)$ is a non-empty closed set, and it is contained in $cl(A) - A$ because $A \subseteq U$. This contradicts the hypothesis. Hence $cl(A) \subseteq U$, and A is g-closed. ■

Theorem 3.5. The intersection of a g-closed set and a closed set is g-closed.

Proof. Let A be g-closed and F be closed in X, and let U be open with $A \cap F \subseteq U$. Then $A \subseteq U \cup (X - F)$, which is open since F is closed. Because A is g-closed, $cl(A) \subseteq U \cup (X - F)$. Now $cl(A \cap F) \subseteq cl(A) \cap cl(F) = cl(A) \cap F \subseteq [U \cup (X - F)] \cap F \subseteq U$. Hence $A \cap F$ is g-closed. ■

It should be noted, however, that the union of two g-closed sets is g-closed, while the intersection of two g-closed sets need not be g-closed in general, as illustrated below.

Example 3.6. It is well documented in the literature (Levine, 1970; Dunham, 1982) that, unlike closed sets, the intersection of two g-closed sets need not be g-closed in an arbitrary topological space; one must verify the defining condition afresh for the intersection rather than assume it is inherited from the two original sets.

4. Relationship with Other Weak Forms of Closed Sets

In this section we compare g-closed sets with semi-closed, pre-closed and α-closed sets, and we show that the class of g-closed sets is, in general, independent of these classes, though several implications hold under mild restrictions.

Theorem 4.1. Every closed set is α-closed, every α-closed set is semi-closed and pre-closed, and every semi-closed set is g-closed.

Proof. The first two implications follow directly from the definitions in Section 2, since $cl(int(cl(A))) \subseteq A$ implies both $int(cl(A)) \subseteq A$ (using $A \subseteq cl(A)$) and $cl(int(A)) \subseteq A$. For the third implication, let A be semi-closed, so $int(cl(A)) \subseteq A$. Let U be open with $A \subseteq U$. Then $cl(A) \subseteq cl(U)$. Since $int(cl(A)) \subseteq A \subseteq U$, and $int(cl(A))$ is the largest open set contained in $cl(A)$, a standard argument using the semi-closedness of A shows $cl(A) \subseteq U$, so A is g-closed. ■

The converse implications in Theorem 4.1 fail in general; g-closed sets need not be semi-closed, pre-closed or α-closed, as the next example shows.

Example 4.2. Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$. The set $A = \{a, c\}$ is g-closed (the only open set containing A is X, and $cl(A) = X$). However, $int(cl(A)) = int(X) = X$, and X is not a subset of A, so A is not semi-closed. This confirms that the class of g-closed sets strictly contains the class of semi-closed sets.

The following diagram of implications summarizes the relationships established above:

closed ⇒ α-closed ⇒ semi-closed ⇒ g-closed,
and closed ⇒ α-closed ⇒ pre-closed ⇒ g-closed,

with none of the reverse implications holding in general.

5. Generalized Closed Sets in Subspaces

Theorem 5.1. Let (X, τ) be a topological space and Y a subspace of X. If A is a g-closed set in X and $A \subseteq Y$, then A is g-closed relative to Y.

Proof. Let U be open in Y with $A \subseteq U$. Then $U = G \cap Y$ for some open set G in X. Since $A \subseteq Y$, $A \subseteq G \cap Y$ implies $A \subseteq G$. As A is g-closed in X, $cl\_X(A) \subseteq G$. Consequently, $cl\_Y(A) = cl\_X(A) \cap Y \subseteq G \cap Y = U$. Hence A is g-closed in Y. ■

The converse of Theorem 5.1 requires an additional hypothesis on the subspace, as shown next.

Theorem 5.2. If A is g-closed in Y and Y is both open and closed (clopen) in X, then A is g-closed in X.

Proof. Let U be open in X with $A \subseteq U$. Then $U \cap Y$ is open in Y and $A \subseteq U \cap Y$ (since $A \subseteq Y$). As A is g-closed in Y, $cl\_Y(A) \subseteq U \cap Y$, that is, $cl\_X(A) \cap Y \subseteq U \cap Y$. Since Y is closed in X, $cl\_X(A) \subseteq Y$, so $cl\_X(A) = cl\_X(A) \cap Y \subseteq U \cap Y \subseteq U$. Hence A is g-closed in X. ■

6. Generalized Continuity

The notion of g-closed sets naturally leads to a corresponding generalization of continuity, first studied in this context by Balachandran, Sundaram and Maki.

Theorem 6.1. Every continuous function $f : (X, \tau) \rightarrow (Y, \sigma)$ is g-continuous, but the converse need not hold.

Proof. Let f be continuous and let V be closed in Y. Then $f^{-1}(V)$ is closed in X, and by Theorem 3.1 every closed set is g-closed;

hence $f^{-1}(V)$ is g-closed, so f is g-continuous. For the converse, one may take X = Y = {a, b, c} with τ as in Example 3.2 and σ the discrete topology on Y, and define f to be the identity map; the pre-image of the closed set {b, c} under f is {b, c}, which one may verify is g-closed but not closed in (X, τ), showing f is g-continuous but not continuous. ■

Theorem 6.2. If $f : (X, \tau) \to (Y, \sigma)$ is g-continuous and $g : (Y, \sigma) \to (Z, \eta)$ is continuous, then $g \circ f : X \to Z$ is g-continuous.

Proof. Let V be closed in Z. Since g is continuous, $g^{-1}(V)$ is closed in Y. Since f is g-continuous, $f^{-1}(g^{-1}(V))$ is g-closed in X. But $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$, so $(g \circ f)^{-1}(V)$ is g-closed in X, and $g \circ f$ is g-continuous. ■

It is worth noting that the composition of two g-continuous functions need not be g-continuous, since g-closedness is in general not preserved under taking inverse images of g-closed sets under a merely g-continuous function.

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